

Embeddings of relatively free groups into finitely presented groups

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1 Introduction

By the Higman embedding theorem [14] every finitely generated recursively presented group can be isomorphically embedded into some finitely presented group. Moreover the embedding can preserve the solvability of the word problem [3] or even the solvability of the conjugacy problem [13]. The embedding can be quasi-isometric [12] and can at the same time dramatically improve the isoperimetric function of the group [2]. Moreover if the word problem for a finitely generated group G is solvable in non-deterministic time $T(n)$ by a Turing machine then G is a quasi-isometric subgroup of a finitely presented group H whose Dehn function is polynomially equivalent to $T(n)$ [2]. Thus the word problem of a finitely generated group G is in NP if and only if G is a quasi-isometric subgroup of a finitely presented group with polynomial Dehn function [2].

There exists (see, for example, Valiev [16]) a relatively simple finite presentation of a group containing all recursively presented groups, but the embedding of a concrete finitely generated group in this universal group is not really effective. The embeddings mentioned in the previous paragraph are not really effective either. In both case one needs to know a precise description of a Turing machine [14] (or a Diophantine equation [16, 9] or an S -machine [2]) “solving” the word problem in the original group in order to find the presentation of the bigger group or to find a copy of the given group inside the universal group.

The main goal of our paper is to show that very often an embedding of G into H can be given in a more explicit and straightforward way. In particular, the description of the set of defining words for H is explicit. (More precisely the S -machines used in these constructions are so small that it is easy to write down all the commands, or simply forget about machines altogether and write down all the relations of the presentation of H .) We can also say a lot about the structure of H , and about the way G is embedded into H .

In this paper, we consider two classes of groups. The first class consists of relatively free groups of finite ranks in varieties of groups. For example, we show that finitely generated free solvable, or free Burnside groups of sufficiently large exponents can be easily embedded into finitely presented groups with polynomial isoperimetric functions. Notice that the isoperimetric functions of these relatively free groups are not even defined because these groups are (as a rule) infinitely presented. In order to deal with relatively free groups, we introduce the

so called *verbal isoperimetric function* of a relatively free group. This function seems to be interesting in itself.

The second class of groups consists of one relator metabelian Baumslag-Solitar groups $BS_{k,1} = \langle a, b \mid a^b = a^k \rangle$. Here x^y stands for $y^{-1}xy$. It is well known that the Dehn function of the group $BS_{k,1}$ is exponential [4]. It is also well known that these groups are very “stubborn”: they “resist” being embedded into groups with small isoperimetric functions. For example, there is a conjecture that the Dehn function of every 1-related group containing $BS_{k,1}$, $k \geq 2$ is exponential. In this paper, we present simple finite presentations of groups with polynomial isoperimetric functions which contain $BS_{k,1}$.

Let us give the necessary definitions. Let H be a group given by a finite presentation $P = \langle x_1, \dots, x_k \mid r_1, \dots, r_\ell \rangle$. A non-decreasing function $f : \mathbf{N} \rightarrow \mathbf{N}$ is called an *isoperimetric function* of this presentation if any word $w = w(x_1, \dots, x_k)$ of length $\leq n$, which is equal to 1 in H , can be written in the free group $F(x_1, \dots, x_k)$ as a product of at most $f(n)$ conjugates of the relators of H . In other words, $f(n)$ is an isoperimetric function of the presentation P if every loop of length $\leq n$ in the Cayley complex corresponding to the presentation P has area at most $f(n)$. Isoperimetric functions of different finite presentations of the same group are *equivalent* in some natural sense [5], so one can talk about isoperimetric functions of the group H forgetting about its presentations. We do not distinguish equivalent isoperimetric functions in this paper. Following Gersten, the smallest isoperimetric function of a group H is called the *Dehn function* of H .

Now let us define the verbal isoperimetric functions. It is easy to see that any variety of groups \mathcal{V} that can be defined by a finite set of identities, can be also defined by a single law $v = 1$ for some word $v = v(x_1, \dots, x_k)$ from the (absolutely) free group F of infinite rank. The verbal subgroup $V \leq F$ consists of all words vanishing in all groups of the variety \mathcal{V} . Any word $w \in V$ is freely equal to a product

$$\prod_{i=1}^N u_i v(X_{i1}, \dots, X_{im})^{\pm 1} u_i^{-1} \quad (1.1)$$

for some words u_i and X_{ij} . We call a non-decreasing function $f_v : \mathbf{N} \rightarrow \mathbf{N}$ a *verbal isoperimetric function* of the word v , if for any word $w \in V$ there is a representation (1.1), where $\sum_{ij} |X_{ij}| \leq f_v(|w|)$. The smallest verbal isoperimetric function will be called the *verbal Dehn function*. Note that the verbal Dehn function exists because in the definition of verbal isoperimetric functions one may restrict oneself to words in variables x_1, \dots, x_n and there are only finitely many such words of any given length.

If an identity $v'(x_1, \dots, x_{m'}) = 1$ is equivalent to the identity $v = 1$ then every value of v' is a product of a fixed number of values of the word v , and vice versa. Thus the verbal Dehn function of the variety V does not depend on the choice of a defining law if one identifies functions which are Θ -equivalent. Recall that two functions $f, g : \mathbf{N} \rightarrow \mathbf{N}$ are Θ -equivalent if there exist positive constants c, d such that for every $n \in \mathbf{N}$ we have:

$$f(n) \leq cg(n) \text{ and } g(n) \leq df(n).$$

Assume that a word w is a product of two words w', w'' which depend on disjoint sets of variables, S' and S'' . Then a representation

$$w = \prod u_i v(X_{i1}, \dots, X_{im}) u_i^{-1}$$

induces similar representations of w' and w'' if we substitute 1 for the variables from S'' or from S' . For the corresponding words X'_{ij} and X''_{ij} one has the obvious inequality $|X'_{ij}| + |X''_{ij}| \leq |X_{ij}|$. Hence the following statement holds.

Proposition 1.1 *Any verbal Dehn function f is superadditive, i.e. $f(n_1 + n_2) \geq f(n_1) + f(n_2)$.*

Remark. Recall that the superadditivity of any (usual) Dehn function is still an open problem (see Guba and Sapir [6]).

In this paper we produce explicit finite presentations of groups $H(v, m)$ to prove the following

Theorem 1.2 *Let $f(n)$ be the verbal isoperimetric function of a group variety \mathcal{V} defined by an identity $v = 1$. Then the free group $F_m(\mathcal{V})$ of rank m in the variety \mathcal{V} can be isomorphically embedded into a finitely presented group $H = H(v, m)$ with an isoperimetric function $n^2 f(n^2)^2$. The presentation of H is explicitly constructed given the word v , and the embedding is quasi-isometric.*

Recall that a finitely generated subgroup G of a finitely generated group H is *quasi-isometric* (in other terminology, undistorted) if there is a positive constant C such that $|g|_G \leq C|g|_H$ for any $g \in G$. Here $|g|_G$, $|g|_H$ are the lengths of g in the fixed finite sets of generators of G and H respectively. This notion is well defined, but the constant C does depend on the choice of the finite generating sets.

Remark. Without the word “explicit” and with slightly worse estimate for the isoperimetric function of the bigger group this theorem is a corollary of the main result of [2]. Indeed, it is easy to see that if f is a verbal isoperimetric function of a group variety \mathcal{V} then the word problem in any relatively free group in this variety can be solved by a non-deterministic Turing machine in time $f(n)$. Thus by the main result in [2], this group can be embedded into a finitely presented group with isoperimetric function $n^2 f(n^2)^4$ (compare with the estimate $n^2 f(n^2)^2$ in our theorem).

For example, let $\mathcal{V} = \mathcal{B}_n$ be the Burnside variety consisting of all groups satisfying the identity $x^n = 1$ for a fixed large odd number n . Then the function $f(s) = s^4$ is a verbal isoperimetric function for \mathcal{V} . One can refer, for instance, to Storozhev’s argument in [11], §28.2: a word w of length s that is equal to 1 in the free Burnside group $B(m, n)$, is a product of n -th powers of some words A_i where the lengths of A_i are bounded by a linear function of s , and the number of factors is bounded by a cubic function. Thus the function s^{18} is an isoperimetric function of $H(v, m)$ in this case. It can be proved by refining Storozhev’s argument that the function $s^{1+\epsilon(n)}$ is a verbal isoperimetric function for \mathcal{B}_n where $\epsilon(n) \rightarrow 0$ for $n \rightarrow \infty$ (R. Mikhajlov, unpublished). Therefore for very large odd n the group $H(x^n, m)$ satisfies a verbal isoperimetric inequality with $f(s) = s^{8+\epsilon}$ for a small $\epsilon > 0$.

The Schreier – Reidemeister rewriting shows by induction, that the variety of all solvable groups of derived length at most d has a polynomial verbal isoperimetric function. It is interesting to calculate (up to the equivalence) the verbal Dehn function of these varieties,

or the varieties of all nilpotent groups of given nilpotency classes, etc. Results of Yu. G. Kleiman [8] show that there exists an identity defining a solvable group variety with non-recursive verbal Dehn function. Indeed, Kleiman [8] constructed a finitely based solvable variety with undecidable identity problem: given a word w , it is impossible to decide whether the law $w = 1$ follows from the defining laws of his variety. Clearly this implies that Kleiman's variety cannot have a recursive verbal isoperimetric function.

Similar questions can be raised for the function which counts just the minimal number of factors N in (1.1). Notice that from the decidability of the Diophantine theory of the free group (Makanin [10]), it follows that this function is recursive if and only if the corresponding verbal Dehn function is recursive.

Theorem 1.3 *The group $BS_{k,1} = \langle a, b \mid b^{-1}ab = a^k \rangle$ is quasi-isometrically embeddable into a finitely presented group $H_{k,1}$ with isoperimetric function n^{10} .*

This theorem gives an upper bound for the Dehn function of an appropriate finitely presented group containing $BS_{k,1}$ as a subgroup, and in contrast to [2], the presentation of $H_{k,1}$ is explicit and S -machines are used only in an implicate way.

2 Defining relations

Let \mathcal{V} be the variety of groups defined by an identity $v(x_1, \dots, x_k) = 1$ where v is a cyclically reduced word, that can be written as $y_1 \dots y_{M-1}$ with $y_s \equiv x_{i_s}^{\pm 1}$.

First we define an auxiliary group $G(v, m)$ which plays the same role as the group $G_N(\mathcal{S})$ in [2] (and bares some similarity with the Boone group from [14]). The set of generators of $G(v, m)$ consists of the letters $a_1, \dots, a_m; q_1, \dots, q_M; k_1, \dots, k_N$ (where $N \geq 29$); $r_{1,1}, \dots, r_{m,k}$.

We present the set of relations of the group $G(v, m)$ in three different ways. First we just list all the relations. Then we will present these relations in terms of an S -machine, and finally we shall present a picture which will contain all the relations, and also show the main corollaries of the relations.

2.A. The list of relations. The finite set of defining relators for $G(v, m)$ is given below by equalities (2.1) - (2.6).

$$r_i^{-1}q_{s+1}r_i = a_jq_{s+1} \quad (2.1)$$

if $i = (j, l)$ and $y_s \equiv x_\ell$;

$$r_i^{-1}q_sr_i = q_sa_j^{-1} \quad (2.2)$$

if $i = (j, l)$ and $y_s \equiv x_\ell^{-1}$;

$$r_i^{-1}q_sr_i = q_s \quad (2.3)$$

for all combinations of i and s which are not considered in (2.1) or (2.2);

$$r_i^{-1}a_jr_i = a_j, \quad i \in I, 1 \leq j \leq m; \quad (2.4)$$

$$r_i^{-1}k_jr_j = k_j, \quad i \in I, 1 \leq j \leq N; \quad (2.5)$$

$$k_1(q_1q_2 \dots q_M) \dots k_N(q_1q_2 \dots q_M) = 1. \quad (2.6)$$

2.B. S -machines. Let us give a precise definition of S -machines [15]. Let k be a natural number. Consider now a *language of admissible words*. It consists of words of the form

$$q_1u_1q_2\dots u_kq_{k+1}$$

where q_i are letters from disjoint sets Q_i , $i = 1, \dots, k+1$, u_i are reduced group words in an alphabet Y_i (Y_i are not necessarily disjoint), the sets $\bar{Y} = \bigcup Y_i$ and $\bar{Q} = \bigcup Q_i$ are disjoint.

Notice that in every admissible word, there is exactly one representative of each Q_i and these representatives appear in this word in the order of the indices of Q_i .

If $0 \leq i \leq j \leq k$ and $W = q_1u_1q_2\dots u_kq_{k+1}$ is an admissible word then the subword $q_iu_i\dots q_j$ of W is called the (Q_i, Q_j) -subword of W ($i < j$).

An S -machine is a rewriting system [7]. The objects of this rewriting system are all admissible words.

The rewriting rules, or S -rules, have the following form:

$$[U_1 \rightarrow V_1, \dots, U_m \rightarrow V_m]$$

where the following conditions hold:

Each U_i is a subword of an admissible word starting with a Q_ℓ -letter and ending with a Q_r -letter (where $\ell = \ell(i)$ must not exceed $r = r(i)$, of course).

If $i < j$ then $r(i) < \ell(j)$.

Each V_i is also a subword of an admissible word whose Q -letters belong to $Q_{\ell(i)} \cup \dots \cup Q_{r(i)}$ and which contains a Q_ℓ -letter and a Q_r -letter.

If $\ell(1) = 1$ then V_1 must start with a Q_1 -letter and if $r(m) = k+1$ then V_m must end with a Q_{k+1} -letter (so tape letters are not inserted to the left of Q_1 -letters and to the right of Q_{k+1} -letters).

To apply an S -rule to a word W means to replace simultaneously subwords U_i by subwords V_i , $i = 1, \dots, m$. In particular, this means that our rule is not applicable if one of the U_i 's is not a subword of W . The following convention is important:

After every application of a rewriting rule, the word is automatically reduced. We do not consider reducing of an admissible word a separate step of an S -machine.

We also always assume that an S -machine is symmetric, that is for every rule of the S -machine the inverse rule (defined in the natural way) is also a rule of this S -machine.

Notice that virtually any S -machine is highly nondeterministic.

Among all admissible words of an S -machine we fix one word W_0 . If an S -machine \mathcal{S} can take an admissible word W to W_0 then we say that \mathcal{S} *accepts* W . We can define the time

function of an S -machine as usual. If $Z \rightarrow Z_1 \rightarrow \dots \rightarrow Z_n = W_0$ is an accepting computation of the \mathcal{S} -machine \mathcal{S} then $|Z| + |Z_1| + \dots + |Z_n|$ is called the *area* of this computation. This allows us to define the *area function* of an S -machine.

In [15], it is showed how to associate a group with any S -machine. Our group $G(v, m)$ is the group associated with the following simple S -machine $\mathcal{S}(v)$.

Its language of admissible words coincides with the set of words of the form $q_1 u_1 q_2 u_2 \dots q_M$ where u_i are any group words in the alphabet $\{a_1, \dots, a_m\}$. Each command of this S -machine corresponds to a variable of v and a letter from $\{a_1, \dots, a_m\}$. Let x be one of the variables of v , which occurs with exponent $+1$ at positions i_1, \dots, i_s of v and occurs with exponent -1 at positions j_1, \dots, j_t and let a be any letter from $\{a_1, \dots, a_m\}$. Then the corresponding command multiplies $q_{i_1+1}, \dots, q_{i_s+1}$ by a on the left, multiplies q_{j_1}, \dots, q_{j_t} by a^{-1} on the right, and does not change other q .

For example, if $v = x^{M-1}$ then v contains only one variable, and commands of the S -machine $\mathcal{S}(v)$ are indexed by letters from $\{a_1, \dots, a_m\}$ and each command has the form $[q_2 \rightarrow a_i q_2, \dots, q_M \rightarrow a_i q_M]$.

Let $W_0 = q_1 \dots q_M$. Here is the main property of the S -machine $\mathcal{S}(v)$. This statement can be easily proved by induction.

The main property of $\mathcal{S}(v)$. An admissible word $q_1 u_1 q_2 \dots u_{M-1} q_M$ is accepted by the S -machine $\mathcal{S}(v)$ if and only if $u_i \equiv u_j \iff y_i = y_j$. In other words acceptable words are obtained from the words of the form $v(u_1, \dots, u_k) = u_{i_1} \cdot \dots \cdot u_{i_{M-1}}$ by inserting q_1, \dots, q_M between the factors u_{i_j} .

For example, if $v = x^{M-1}$ then accepted words have the form $q_1 u q_2 u \dots u q_M$ where u is an arbitrary word in the alphabet $\{a_1, \dots, a_n\}$.

The construction (essentially from [15]) which simulates an S -machine in a group is the following.

Let Θ be the set of rules of \mathcal{S} . Let us call one of each pair of mutually inverse rules from Θ *positive* and the other one *negative*. The set of all positive rules will be denoted by Θ_+ .

Let N be any positive integer (≥ 29). Let A be the set of all letters occurring in the admissible words of \mathcal{S} union with the set $\{k_j \mid j = 1, \dots, N\} \cup \Theta_+$.

Our group is generated by the set A subject to the set $\mathcal{P}_N(\mathcal{S})$ of relations described below.

1. Transition relations. These relations correspond to elements of Θ_+ .

Let $r \in \Theta_+$, $r = [U_1 \rightarrow V_1, \dots, U_p \rightarrow V_p]$. Then we include relations $U_1^r = V_1, \dots, U_p^r = V_p$ into $\mathcal{P}_N(\mathcal{S})$. If some set Q_j does not have a representative in any of the words U_i then we include all the commutativity relations $q^r = q$, $q \in Q_j$.

2. Auxiliary relations.

These are all possible relations of the form $rx = xr$ where x is one of the letters in $\{a_1, \dots, a_m, k_1, \dots, k_N\}$, $r \in \Theta^+$.

3. The hub relation.

$$k_1 W_0 k_2 W_0 \dots k_N W_0 = 1$$

It is easy to see that the group presentation associated with our S -machine $\mathcal{S}(v)$ coincides with the presentation constructed above in section A.

2.C. A picture. For simplicity, let us take $v = x^n$: our construction does not depend much on the word v anyway.

Further simplifying the situation (and the future picture) let us take $n = 3$. The construction really does not depend much on n either, so we shall sometimes write n instead of 3. Figure 1 shows the van Kampen diagram (below it will be called a *disc*) with boundary label $\Sigma(q_1 u q_2 u q_3 u q_4) = k_1 q_1 u q_2 u q_3 u q_4 k_2 q_1 u q_2 u q_3 u q_4 k_3 \dots k_N q_1 u q_2 u q_3 u q_4$.

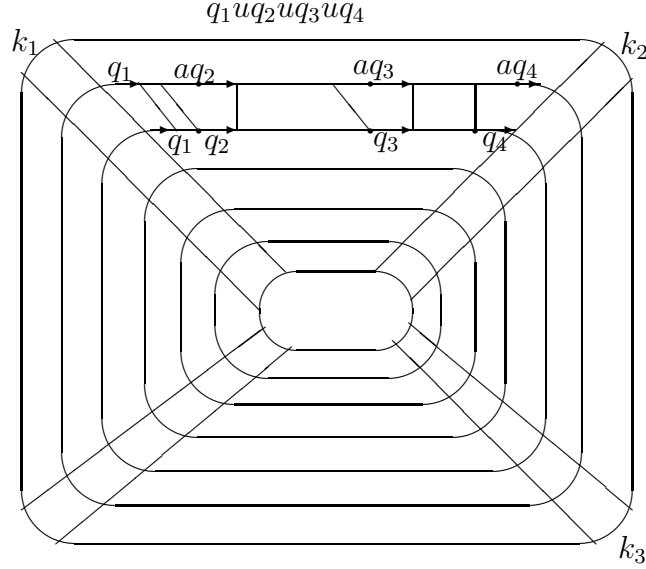


Fig. 1.

On the boundary of this diagram we can read the word $\Sigma(q_1 u q_2 u q_3 u q_4)$. The words on each of the concentric circles is labeled by $\Sigma(q_1 u_i q_2 u_i q_3 u_i q_4)$ where u_i is a prefix of u of length $i - 1$. The word written on the innermost circle is the hub, $\Sigma(q_1 q_2 q_3 q_4)$. The edges connecting the circles are labeled by letters r_1, \dots, r_m corresponding to the letters of u . The cells tessellating the space between the circles have labels

- $q_i^{r_j} = a_j q_i, i = 2, 3, 4, j = 1, \dots, m; q_1^{r_j} = q_1$.
- $ar = ra, a \in \{a_1, \dots, a_m\}, r \in \{r_1, \dots, r_m\}$
- $kr = rk, k \in \{k_1, \dots, k_N\}, r \in \{r_1, \dots, r_m\}$.

These are exactly the relations of our group $G(v, m)$.

The following two lemmas summarize two main features of the presentation of the group $G(v, m)$. For any reduced words X_1, \dots, X_k in the alphabet $a_1^{\pm 1}, \dots, a_m^{\pm 1}$ we define the word $\Lambda(X_1, \dots, X_k)$ to be the word $u(X_1, \dots, X_k)$ with letters q_1, \dots, q_M inserted between factors X_{i_j} (recall that by the main property of $\mathcal{S}(v)$ these are all acceptable words of the S -machine $\mathcal{S}(v)$.) More precisely

$$\Lambda(X_1, \dots, X_k) \equiv q_1 X_{i_1} q_2 X_{i_2} \dots X_{i_{M-1}} q_M,$$

provided $v(x_1, \dots, x_k) \equiv x_{i_1} x_{i_2} \dots x_{i_{M-1}}$. The following claim is an immediate corollary of the relations (2.1)-(2.5) (and is evident from Figure 1).

Lemma 2.1 *Assume $i = (j, \ell)$. Then in view of relations (2.1) - (2.4), the word $r_i^{-1}\Lambda(X_1, \dots, X_k)r_i$ (the word $r_i\Lambda(X_1, \dots, X_k)r_i^{-1}$) is equal to the word $\Lambda(X'_1, \dots, X'_k)$, where in the free group $X'_u = X_u$ for $u \neq \ell$, and $X'_\ell = X_\ell a_j$ ($X'_\ell = X_\ell a_j^{-1}$).*

□

Now set

$$\Sigma(X_1, \dots, X_k) \equiv k_1\Lambda(X_1, \dots, X_k) \dots k_N\Lambda(X_1, \dots, X_k).$$

In particular $\Sigma(1, \dots, 1)$ is the left-hand side of the relation (2.6). Since any k -tuple (X_1, \dots, X_k) is a result of iterated multiplications of the components of the k -tuple $(1, \dots, 1)$ by letters $a_j^{\pm 1}$, Lemma 2.1 and relations (2.6) imply

Lemma 2.2 *For any reduced words X_1, \dots, X_k , the word $\Sigma(X_1, \dots, X_k)$ is conjugate to the word $\Sigma(1, \dots, 1)$ in virtue of relations (2.1) - (2.6). In particular, $\Sigma(X_1, \dots, X_k) = 1$ in the group $G(v, m)$.*

□

This lemma is also evident from Figure 1, because Figure 1 is the van Kampen diagram over the presentation of $G(v, m)$ with boundary label

$$\Sigma(X_1, \dots, X_k).$$

Now let us define the presentation of the group $H = H(v, m)$ (the construction is similar to the Aanderaa construction from [1]). Let us add new letters ρ, d, b_1, \dots, b_m to the above presentation of $G(v, m)$, and add the following relations:

$$\rho^{-1}k_1\rho = k_1d^{-1}, \quad \rho^{-1}k_2\rho = dk_2; \tag{2.7}$$

$$\rho^{-1}k_j\rho = k_j, \quad 3 \leq j \leq N; \tag{2.8}$$

$$\rho^{-1}q_j\rho = q_j, \quad 1 \leq j \leq M; \tag{2.9}$$

$$\rho^{-1}a_j\rho = a_j, \quad 1 \leq j \leq m; \tag{2.10}$$

$$d^{-1}a_jd = a_jb_j, \quad 1 \leq j \leq m; \tag{2.11}$$

$$d^{-1}q_jd = q_j, \quad 1 \leq j \leq M; \tag{2.12}$$

$$b_ja_\ell = a_\ell b_j, \quad 1 \leq j, \ell \leq m; \tag{2.13}$$

$$b_jq_\ell = q_\ell b_j, \quad 1 \leq j \leq m, 1 \leq \ell \leq M; \tag{2.14}$$

$$w(b_1, \dots, b_m) = 1 \quad (2.15)$$

for any cyclically reduced word $w = w(b_1, \dots, b_m)$ which is equal to 1 in the relatively free group $F_m(\mathcal{V})$ with the basis (b_1, \dots, b_m) .

It is easy to see that these relations do not depend on the structure of the word v (only on the length of v). All these relations can be put in one picture, the following Figure 2. These relations together with the relations of $G(v, m)$ form the presentation of $H(v, m)$.

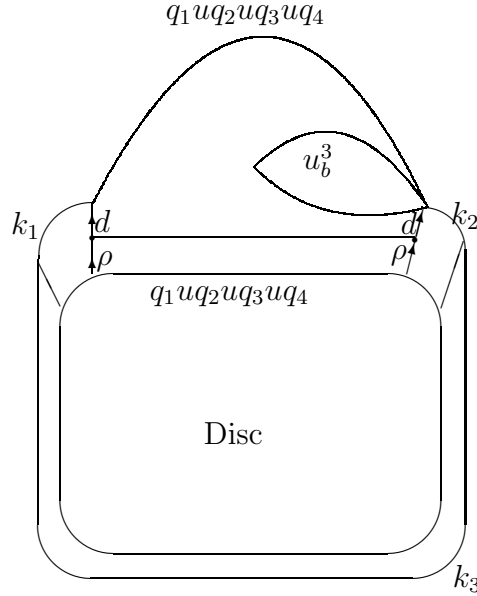


Fig. 2.

This is an annular diagram over the presentation of $H(v, m)$ (as above we assume for simplicity that $v = x^3$). It is obtained in the following way. Take the disc Δ on Figure 1. Since ρ commutes with all generators of $G(v, m)$ except k_1 and k_2 , and $k_1^\rho = k_1 d^{-1}$, $k_2^\rho = d k_2$, we can form an annulus of ρ -cells with the inner boundary labeled by the same word as the boundary of Δ , and the outer boundary labeled by the same word with d^{-1} inserted next to the right of k_1 and d inserted next to the left of k_2 . Glue in the disc Δ inside this annulus. We obtain the part of the diagram on Figure 2 formed by the disc and the ρ -annulus enveloping the disc. Let us call this part Δ_1 . Now d commutes with all q 's, and we have that $a_i^d = a_i b_i$. Also take into account that b_i commutes with all the a 's and q 's. This implies that if $U = q_1 u q_2 u q_3 u q_4$ is the word written between k_1 and k_2 on the boundary of Δ (read clockwise) then

$$U^d = q_1 u u_b q_2 u u_b q_3 u u_b q_4 = q_1 u q_2 u q_3 u q_4 u_b^3 = U u_b^3$$

(all equalities hold modulo the presentation of $H(v, m)$). Here u_b is the word u rewritten in the alphabet $\{b_1, \dots, b_m\}$. The corresponding diagram over $H(v, m)$ can be attached to Δ_1 along the arc labeled by $d^{-1} W d$. Let the resulting diagram be denoted by Δ_2 . Now we can

get the diagram on Figure 2 by identifying the ends of the arc labeled by u_b^3 on the boundary of Δ_2 .

Notice that the outer boundary of the diagram on Figure 2 is labeled by the same word as the boundary of the disc Δ . Thus all the relations $u_b^3 = 1$ (relations (2.15) above) follow from the other relations from the presentation of $H(x^3, m)$. The next statement generalizes these observations to an arbitrary v .

Lemma 2.3 *Let X_1, \dots, X_k be words in $a_1^{\pm 1}, \dots, a_m^{\pm 1}$, and assume that Y_1, \dots, Y_k are their copies in $b_1^{\pm 1}, \dots, b_m^{\pm 1}$ (obtained by replacing every a_j by b_j in X_1, \dots, X_k). Then the relation*

$$d^{-1}\Lambda(X_1, \dots, X_k)d = \Lambda(X_1, \dots, X_k)v(Y_1, \dots, Y_k)$$

follows from relations (2.11) - (2.14); relations (2.15) follows from relations (2.1) - (2.14). In particular, H is a finitely presented group.

□ The first claim is an immediate corollary of the relations (2.11) – (2.14) and the definition of the word $\Lambda \equiv \Lambda(X_1, \dots, X_k)$. As for the second statement, it suffices to prove it for all words of the type $w(b_1, \dots, b_m) \equiv v(Y_1, \dots, Y_k)$ only. In order to do that, we will apply Lemma 2.2, relations (2.7) – (2.10), and the first claim of Lemma 2.3:

$$\begin{aligned} 1 &= \rho^{-1}\Sigma(X_1, \dots, X_k)\rho = k_1d^{-1}\Lambda dk_2\Lambda \dots k_N\Lambda = \\ & k_1\Lambda v(Y_1, \dots, Y_k)k_2\Lambda \dots k_N\Lambda. \end{aligned}$$

By Lemma 2.2 the last product remains being equal to 1 after erasing the factor $v(Y_1, \dots, Y_k)$. Hence this factor vanishes itself. □

3 Bands and annuli.

Consider a simply connected van Kampen diagram Δ over the presentation of the group H (see [9] or [11]; we assume that any edge of any van Kampen diagram is labeled by one letter, as in [11]). If a face Π of Δ corresponds to a relation containing letters x and y then Π is said to be a (x, y) -cell. Thus we can talk about (ρ, a) -cells, (a_j, ρ) -cell, (r, q) -cells, etc. Similarly if the relation contains letter x then we shall call the corresponding cell an x -cell.

The boundary of a ρ -cell Π has exactly two ρ -edges labeled by $\rho^{\pm 1}$. These labels are inverses of each other when one reads the boundary label of Π . This gives us an opportunity to construct “bands” of several ρ -cells.

A ρ -band of length 0 has no faces and consists of one ρ -edge. A ρ -band of length 1 is just a single ρ -cell. Assume by induction, that we have a ρ -band $T' = [\Pi_1, \dots, \Pi_{s-1}]$ of length $s - 1$ constructed of $s - 1$ distinct ρ -cells Π_1, \dots, Π_{s-1} , and the boundary of T' has a ρ -edge e , which is a common edge of the boundary $\partial(\Pi_{s-1})$ and the boundary of a ρ -cell Π_s which is distinct from Π_1, \dots, Π_{s-1} . Then we are able to construct a ρ -band $T = [\Pi_1, \dots, \Pi_s]$ of length s whose boundary ∂T is the union of the the boundaries $\partial T'$ and $\partial \Pi_s$ minus the edge e . A ρ -band T is *maximal* if is not contained in a ρ -band of a greater length.

Thus, the boundary of a ρ -band T has the form $e_1 p e_2 q$, where e_1 and e_2 are ρ -edges (we call them *ends* of T), and the paths p, q (*sides* of the band T) consist of a - and q -edges, but contain no ρ -edges.

If the ends of T coincide, one may identify them and the annular subdiagram T is called a ρ -annulus. For example the diagram on Figure 2 contains a ρ -annulus enveloping the disc.

Similarly we define d -bands (and annuli) that by definition can be constructed of (d, a) -, (d, t) -, and (d, q) -cells.

b -bands can be constructed of (b, a) - and (b, q) -cells.

r -bands are constructed of (r, q) -, (r, a) -, and (r, k) -cells.

q -bands are constructed of (q, r) -, (q, ρ) -, (q, d) -, and (q, b) -cells.

a -bands are created of (a, r) -, (a, ρ) -, (a, d) -, and (a, b) -cells.

Notice that (ρ, d) -cells of type (2.7) cannot be included in a d -band but they can be *terminal* for d -bands, i.e. a maximal d -band can end only on the contour of a (ρ, k) -cell or on the contour of the diagram Δ . Similarly, (r, q) -cells are terminal for a -bands, *hubs*, corresponding to relation (2.6), are terminal for k - and q -bands, (d, a) -cells are terminal for b -bands. Also a b -band can terminate on the contour of a G_b -cell (by definition, a G_b -cell corresponds to a relation (2.15)).

Now consider an r -band $T = [\pi_0, \pi_1, \dots, \pi_\ell, \pi_{\ell+1}]$ and a q -band $T' = [\pi_0, \gamma_1, \dots, \gamma_s, \pi_{s+1}]$ which have no common faces except for π_0 and $\pi_{\ell+1}$, and with all ends of T and T' lying on the outer boundary of the annulus S formed by T and T' . Then this annulus is called an (r, q) -annulus. It consists of the r -part T and the q -part T' . The faces π_0 and $\pi_{\ell+1}$ are its *corner* cells.

The definitions of (ρ, a) -, (q, b) -annuli, etc. are quite similar.

A diagram Δ is called *minimal* in this section if there exists no other diagram Δ' such that (1) Δ' has the same boundary label as Δ , and (2) the number of faces of each of the types (2.1) - (2.15) in Δ' does not exceed the similar number for Δ , (3) the total number of faces in Δ' is smaller than the number of faces in Δ . In the next two sections we shall a stronger definition of minimality.

The main lemma of this section claims that there are no annuli of various kinds in minimal diagrams without hubs.

Lemma 3.1 *Let Δ be a minimal diagram over H containing no hubs. Then Δ has no*

- (1) ρ -annuli,
- (2) r -annuli,
- (3) (r, q) -annuli,
- (4) q -annuli,
- (5) (r, k) -annuli,
- (6) k -annuli,
- (7) (ρ, k) -annuli,
- (8) (a, b) -annuli,
- (9) (d, a) -annuli,
- (10) (ρ, a) -annuli,
- (11) (r, a) -annuli,
- (12) d -annuli,

- (13) b -annuli,
- (14) a -annuli,
- (15) (ρ, q) -annuli,
- (16) (d, q) -annuli,
- (17) (q, b) -annuli

□ To prove statements (1) - (17) we use a simultaneous induction on the number of faces in the minimal subdiagram Δ_S containing a conjectural counterexample, i.e. an annulus S . This means that we may assume that Δ has no annulus S' of types (1) - (17) such that the subdiagram $\Delta_{S'}$ has fewer faces than Δ_S .

(1) Let S be a ρ -annulus. Assume that S has a (ρ, k_j) -cell. Then this cell belongs to a k_j -band T , which must intersect S at least twice, because by the lemma condition Δ contains no hubs (terminal cells for k -bands). In such a case T and a subband of S form a smaller (ρ, k) -annulus S' than S , contrary to claim (7) of the lemma. The only case when S' is not smaller than S is when S consists just of two (ρ, k_j) -cells with a common ρ -edge and with “mirror” labels. Such a pair of mirror faces is impossible in a minimal diagram. For more details on the cell cancellation, see [9].

Therefore S contains only (ρ, a) - and (ρ, q) -cells. Consequently the outer and the inner boundaries of S have identical labels. This makes it possible to delete the interior of S and then identify the sides of S . Such a surgery does not change the boundary label of Δ , contrary to the minimality of Δ .

(3) Assume that S is a (r, q) -annulus. Let T be the q -part and T' be the r -part. S has no k -cells, because otherwise a smaller (r, k) -annulus appears, contradicting claim (5). Analogously, by (3) (for smaller annuli) there are no non-corner (r, q) -cells in S . The same argument shows that there are no other (r, q) -cells in the subdiagram Δ .

The corner (r, q) -cells are included in the same r - and q -bands. This implies that they correspond to the same relation and simultaneously have or have no a_j -edges (for the same j) on the inner border of S with opposite directions. Only these corner cells can be terminal for a -bands crossing S .

Therefore if there exist non-corner cells in S , then by (11) there exist exactly 2 such cells, they must be neighbors in the r -part of S , and must have “mirror” labels, contrary to the minimality of Δ .

Hence the r -part of S has no non-corner cells. Then the standard cancelation argument can be applied to the corner cell. This contradicts the minimality assumption again.

(2), (4) - (17) The proofs of all these statements are similar to the two proofs of (1) and (3) given above. The reader could examine them as an exercise or read similar explanation for Lemma 6.1 [12] (claims (1) - (20)) or Section 7 of [15]. □

4 Hubs and spokes

A *spoke* is a maximal k -band having an end on a hub (or on a disc in the next section). Obviously, another end lies on a hub too, or on the boundary $\partial\Delta$.

It will be convenient to restrict the notion of a minimal diagram used in previous section as follows.

A *type* $\tau = \tau(\Delta)$ of a diagram Δ over H is the 4-tuple $\tau = (\tau_1, \tau_2, \tau_3, \tau_4)$ where τ_1 is the number of hubs in Δ (or, in the next section, the number of discs), τ_2 is the number of (ρ, k) - and (r, k) -cells, τ_3 is the number of all other faces except (b, q) -, and (b, a) - cells, τ_4 is the number of (b, t) -, (b, q) -, and (b, a) -cells in Δ . Set $\tau < \tau'$ if $\tau_1 < \tau'_1$, or $\tau_1 = \tau'_1$, but $\tau_2 < \tau'_2$, and so on. Further a diagram is said to be *minimal* if it has the minimal type among all diagrams with the same boundary label. Clearly, this notion of minimality is stronger than that in Section 3, that is a diagram which is minimal under this definition is also minimal under the definition in the previous section.

Since our construction of the group $G(v, m)$ is essentially the same as the construction of the group $G_N(\mathcal{S})$ of [15], the next lemma follows from Lemma 11.1 of [15]. Nevertheless we present a direct proof here.

Lemma 4.1 *Let Δ be a minimal diagram over $G(v, m)$ containing two hubs Π_1 and Π_2 . Assume that the hubs have two common consecutive spokes T_1 and T_2 such that the subdiagram Δ_0 bounded by these spokes and the hubs contains no hubs. Then Δ is not minimal diagram: by passing to another diagram with the same boundary label one can decrease the number of hubs by 2.*

□ Let Δ_1 be the subdiagram constructed of Δ_0 and the spokes T_1, T_2 . Without loss of generality we may assume that Δ_1 is a minimal diagram. By Lemma 3.1 (5) for Δ_1 , every r -band crossing T_1 , must cross T_2 as well. Therefore each of the 4 sides of the bands T_1, T_2 must have the same label $V = V(r_1, r_2, \dots)$. Thus the boundary label of Δ_0 is the commutator $[V, \Lambda_1]$ for the word $\Lambda_1 \equiv \Lambda(1, \dots, 1) \equiv q_1 t q_2 t \dots q_M$, i.e. V commutes with Λ_1 modulo all the defining relations of $G(v, m)$ excluding the hub. Being a word in the alphabet $r_1^{\pm 1}, r_2^{\pm 1}, \dots$, V commutes with letters k_j (see relations (2.5)). Consequently, V commutes (modulo (2.1) - (2.5)) with the word W , which is a cyclic permutation of the left-hand side of the hub relation (2.6) written on Π_1, Π_2 in opposite directions starting with the ends of the band T_1 .

Thus, if we cut out the hubs Π_1, Π_2 from Δ and then make a cut along the band T_1 border, we get a hole labeled by the word $[V, W]$ equal to 1 by (2.1) – (2.5). Now by the van Kampen lemma we are able to insert a diagram of a type $(0, \star, \star, \star)$ in this hole, reducing the number of hubs in Δ by 2. □

Lemma 4.2 *Let Δ be a diagram over $H(v, m)$ containing two hubs Π_1, Π_2 with common consecutive k_j - and k_s -spokes T_1 and T_2 where $\{j, s\} \neq \{1, 2\}$. Assume that the subdiagram Δ_0 bounded by these spokes and the hubs contains no hubs. Then the diagram Δ is not minimal: By passing to another diagram with the same boundary label, one can decrease the number of hubs by 2.*

□ The proof is similar to the proof of Lemma 4.1, but now the word V may contain $\rho^{\pm 1}, r_1, r_2, \dots$. So, the difference is that V may not commute with the letters k_1, k_2 occurring

in W . However k_1, k_2 occur just in the subword $k_1\Lambda_1k_2$ of the left-hand side in (2.7), which is also a subword of W in view of the condition $\{k_j, k_s\} \neq \{1, 2\}$. Therefore it suffices to check that V commutes with k_ℓ for $\ell \neq 1, 2$, with Λ_1 , and with $k_1\Lambda_1k_2$ modulo relations (2.1) - (2.5), (2.7) - (2.15). The first property follows from (2.5) and (2.9). The second one can be explained exactly as in Lemma 4.1.

To prove the third commutativity, notice first of all that by relations (2.15) and Lemma 2.3 d commutes with $\Lambda(X_1, \dots, X_k)$ for any words X_1, \dots, X_k in the alphabet $\{a_1^{\pm 1}, \dots, a_m^{\pm 1}\}$. Then ρ commutes with both $\Lambda(X_1, \dots, X_k)$ (see (2.9), (2.10)) and $k_1\Lambda(X_1, \dots, X_k)k_2$ since by (2.7) $\rho^{-1}k_1 = k_1d^{-1}\rho^{-1}$ and $k_2\rho = \rho dk_2$.

Also recall that by (2.5) and Lemma 2.2

$$r_i^{\pm 1}k_1\Lambda(X_1, \dots, X_k)k_2r_i^{\mp 1} = k_1r_i^{\pm 1}\Lambda(X_1, \dots, X_k)r_i^{\mp 1}k_2 = k_1\Lambda(X'_1, \dots, X'_k)k_2$$

for some X'_1, \dots, X'_k .

Therefore permuting the words V^{-1} and V (these are word in $\rho^{\pm 1}, r_1^{\pm 1}, r_2^{\pm 1}, \dots$) letter-by-letter with k_1 and k_2 , we get

$$V^{-1}k_1\Lambda_1k_2V = k_1V^{-1}\Lambda_1Vk_2.$$

The right-hand side is equal to $k_1\Lambda_1k_2$, as was mentioned earlier. \square

With any minimal diagram Δ over $H(v, m)$ we associate the following graph $\Gamma = \Gamma_\Delta$. One vertex of Γ (*exterior* vertex) is taken outside Δ on the plane. Every *interior* vertex is chosen inside a hub. The edges between the vertices are drawn along the "medians" of the spokes. The *exterior* edges are incident to the exterior vertex. The other edges are *interior*. Finally to complete the definition of Γ , we erase the interior k_1 -spoke for every pair of hubs connected by both k_1 - and k_2 -spokes.

By Lemma 4.2 there are no bigons formed by interior edges of Γ . Also there are no loops in Γ because every letter k_j occurs once in the left-hand side of (2.7). Therefore in standard way the Euler formula implies that Γ has many exterior edges. (The restriction $N - 1 \geq 6$ would be enough here; the stronger condition $N - 1 \geq 28$ will be useful for Sections 5-7.) The following statement (see Lemma 2.13 in [2], or Lemmas 3.2, 3.3 in [12], or Lemma 11.5 in [15]) will be sufficient for our purpose.

Lemma 4.3 *Let Δ be a minimal diagram containing at least one hub. Then there exists a hub Π in Δ such that at least $N - 4$ consecutive spokes starting on Π , have their ends on the boundary $\partial\Delta$, and moreover there are no other hubs between the spokes of this set. The number of k -edges in $\partial\Delta$ is at least 3 times greater than the number of hubs in Δ .*

\square

Lemma 4.4 *The natural homomorphism of the group $F_m(\mathcal{V})$ onto $H(v, m)$ (well defined in view of relations (2.15)) is injective.*

□ Assume that a word $w \equiv w(b_1, \dots, b_2)$ vanishes under the homomorphism. Then there exists a minimal van Kampen diagram Δ with the boundary label w . By Lemma 4.3 there are no hubs in Δ because no letter k_j occurs in w . Then by Lemma 3.1 (1) Δ has no ρ -annuli, and consequently, it has no ρ -cells at all. Quite similarly, Lemma 3.1 allows us to exclude r -, q -, k -, d -, and a -cells from Δ consequently. For example, there are no d -cells because maximal d -bands could terminate on ρ -cells only. Thus, Δ has G_b -cells only, that correspond to relations (2.15). Hence $w = 1$ in G as desired. □

5 The band structure of disc-based diagrams

It will be convenient to extend the list of relations of the group $H(v, m)$ by adding the relations $\Sigma(X_1, \dots, X_k) = 1$ for all k -tuples of reduced words X_1, \dots, X_k in $a_1^{\pm 1}, \dots, a_m^{\pm 1}$. Such an enlargement does not change $H(v, m)$ by Lemma 2.2. Any face of a diagram over this presentation of $H(v, m)$, that corresponds to some relation $\Sigma(X_1, \dots, X_k) = 1$, will be called a *disc*.

The notion of a minimal diagram will be further restricted by replacing discs for hubs in the definition of a minimal diagram from the previous section. With any minimal diagram Δ we associate a graph $\Gamma(\Delta)$. Its definition repeats the definition of the graph Γ_Δ given in Section 4 where discs replace hubs. Notice that by Lemma 2.2 every disc can be replaced by a hub (which is a disc too) and a number of faces of smaller ranks (but the resulted diagram may be not minimal). The possibility of such a replacement and Lemma 4.2 show that the graph $\Gamma(\Delta)$ has no bigons as well. Therefore the statement of Lemma 4.3 is also true for $\Gamma(\Delta)$.

Let S be a ρ - or r -band that consequently intersects at (ρ, k) - or (r, k) -cells a series of consequent spokes T_1, \dots, T_ℓ starting on a disc D . We say that the band S *envelopes* disc D if $\ell > N/2$, and there are no other discs in the sectors formed by S , T_j and T_{j+1} for $j = 1, \dots, \ell - 1$.

Lemma 5.1 *A minimal diagram Δ over $H(v, m)$ has no bands which envelope discs.*

□ The proof is completely similar to the proofs of Lemma 8.4 [12] or Lemma 2.17 in [2]. Therefore we give just a brief explanation below referring for details to [12] or [2].

Arguing by contradiction, one can choose the closest to D band S that envelopes it. Then the intersection cells π_1, \dots, π_ℓ of S and T_1, \dots, T_ℓ have common k -edges with D . It suffices to prove that the diagram Δ_0 consisting of D and π_1, \dots, π_ℓ and considered separately from Δ , is not minimal.

For this purpose we attach auxiliary (r_s, k) -cells $\pi_{\ell+1}, \dots, \pi_N$ to Δ_0 along the $N - \ell$ free k -edges of the disc D (if S is a r_s band) so that all the cells π_1, \dots, π_N would be attached to D uniformly, i.e. their r -edges would be directed all 'to' or all 'from' D . Then adding several faces of smaller ranks we can get a diagram Δ_1 with a label $\Sigma(X'_1, \dots, X'_k)$ by Lemma 2.2.

Therefore, conversely, one can construct a diagram Δ_2 , with the same boundary label as Δ_0 , consisting of a disc D' labeled by $\Sigma(X'_1, \dots, X'_k)$, mirror copies of (k, r) -cells $\pi_{\ell+1}, \dots, \pi_N$, and faces of smaller ranks. But this contradicts to the minimality of Δ_0 since $N - \ell < \ell$.

If S is a ρ -band, the proof is similar, but the boundary label of the disc D' coincides with the label of D , since ρ commutes with $\Sigma(X_1, \dots, X_k)$ as was explained in Lemma 4.2. \square

Lemma 5.2 *A minimal diagram Δ has no annuli of types (1) - (17) from Lemma 3.1 (even if hubs or discs occur in Δ).*

\square This is similar to Lemmas in Section 4 of [2]. Let us show, for example, that statement (3) of this lemma can be deduced from statement (3) of Lemma 3.1.

Let Δ_S be a minimal subdiagram of Δ , containing a (r, q) -annulus S . By Lemma 3.1(3) it contains a disc. By Lemma 4.3 there exists a disc D in Δ_S such that its consecutive spokes T_1, \dots, T_{N-4} intersect the r -part R of S (since the q -part has no k -cells at all). If there are other discs in Δ_S and the q -part Q of S occurs between some T_j and T_{j+1} , then again as in Lemma 4.3 (but for the graph obtained by erasing the vertex in D and the edges incident to it), we get another disc D' and spokes T'_1, \dots, T'_{N-5} starting on it, such that R intersects them consecutively, and there are neither hubs nor Q between the spokes. But this contradicts Lemma 5.1 because $N - 5 \geq N/2$. \square

Lemma 5.3 *Any two distinct maximal bands T and T' have at most one common face in a minimal diagram over $H(v, m)$.*

\square Basically the statement follows from Lemma 5.2. However we have to remember that a-priori, a multiple intersection of two bands T and T' does not imply that they form even one annulus, because one or both ends of the band can be inside the “annulus”. Figure 3 shows a spiral multiple intersection.

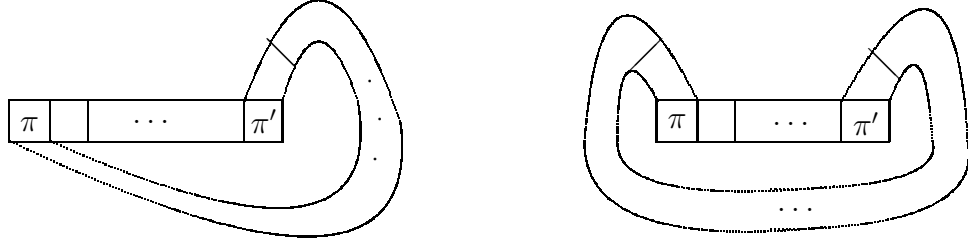


Fig. 3.

In fact such spirals cannot occur, and the reader can find details in Lemmas 5.1 and 5.8 from [2]. Here we just explain the idea for the particular case when T is a d -band and T' is an a -band. In this case a terminal cell Π for T must be inside the subdiagram Δ_S bounded by S . Π is a (ρ, k) -cell (see relations (2.7)). Therefore by Lemma 4.3 and Lemma 3.1 (6) the boundary $\partial\Delta_S$ must be crossed by at least one k -band. But both d -band T and a -band T' have no k -cells at all, a contradiction. \square

The following statement is similar to Lemma 4.36 in [2].

Lemma 5.4 *There is no b -band T in a minimal diagram Δ such that both ends of T belong to G_b -cells.*

□ First assume that both ends of T belong to the boundary of one G_b -cell Π . Then the boundary of T and a subpath of $\partial\Pi$ form the boundary of a subdiagram Δ_0 with the boundary label UV where U is a word in $b_1^{\pm 1}, \dots, b_m^{\pm 1}$, and V is a word in $a_1^{\pm 1}, \dots, a_m^{\pm 1}, q_1^{\pm 1}, \dots, q_M^{\pm 1}$. By lemmas 4.3 and 3.1 Δ_0 has no discs and k -cells. By lemma 3.1 the band T has no a - and q -cells because otherwise we would get (a, b) - or (q, b) -bands. Thus the band T has length 0. Hence there is a loop in $\partial\Pi$ whose label must be equal to 1 in the group G by Lemma 4.4. Then one can replace the interior of this loop and Π by a single G_b -cell, contrary to the minimality of Δ .

Assume now that the ends of T belong to distinct G_b -cells Π_1 and Π_2 . Recall that the boundary label of T commutes with any word in $b_1^{\pm 1}, \dots, b_m^{\pm 1}$ by relations (2.13) and (2.14). Therefore the diagram Δ is not minimal: the subdiagram consisting of the two G_b -cells connected by the border of T , can be replaced by a diagram consisting of one G_b -cell and several cells of smaller ranks that correspond to relations (2.13) and (2.14). □

6 Upper bounds for the number of cells in minimal diagrams

Recall that by Lemma 2.3 the group $H(v, m)$ has a finite presentation.

Lemma 6.1 *For any word w of length n in the generators of the group $H(v, m)$, that is equal to 1 in $H(v, m)$, there is a diagram over the finite presentation (2.1) - (2.14) with $n^2 O(f(O(n^2))^2)$ cells, where f is the function given in Theorem 1.*

□ First consider a minimal diagram Δ with the boundary label w over the disc-based presentation.

It follows from the statement of Lemma 4.3 for discs, that the number of discs in Δ is $O(n)$. Therefore by Lemma 5.2 the number of maximal ρ -, r -, q -, and k -bands in Δ is $O(n)$ because the boundary of any disc has no ρ - and r -edges, and has $O(1)$ q -, and k -edges.

Therefore by Lemma 5.3 the number of (r, q) -cells (the intersections of r - and q -bands) is $O(n^2)$. Similarly, the number of (r, k) -, (ρ, k) -, and (ρ, q) -cells is $O(n^2)$.

By Lemma 5.2 the number of maximal d -bands in Δ is $O(n^2)$ since only (ρ, k) -cells can be terminal for d -bands.

A similar argument shows that the number N_1 of all maximal a -bands terminating on (r, q) -cells or on $\partial\Delta$ is $O(n^2)$. To obtain similar upper bound for the number of all a -bands, it suffices to explain why the number N_2 of maximal a -bands having both ends on discs, is not greater than N_1 . Indeed the number of common spokes of two discs is at most 2. Consequently the number of common a -bands for 2 discs is at most $3/N < 1/9$ of the number of maximal a -bands starting on each of the disc. This implies that $N_2 \leq N_1$.

The upper bounds for the numbers of maximal k -, q -, and a -bands show that the sum of perimeters of all discs in Δ is $O(n^2)$.

Now by Lemma 5.3 we are able to conclude that the number of (ρ, a) -, (r, a) -, (d, q) -, (d, t) -cells is $O(n^3)$, and the number of (d, a) -cells is $O(n^4)$.

Only (d, a) -cells can be terminal for b -bands. So the above estimate of the number of (d, a) -cells together with Lemma 5.2 imply that the number of maximal b -bands is $O(n^4)$. This implies, first of all, that the number of (b, a) -cells is $O(n^6)$, and the number of (b, q) - and (b, t) -cells is $O(n^4)$. Second, this implies that the sum of perimeters of all G_b -cells is $O(n^4)$. By the definition of the function f every G_b -cell with boundary length $\leq s$ can be tiled by cells with boundary labels of the form $v(Y_1, \dots, Y_k)$ and with $\sum |Y_{ij}|$ at most $f(s)$. By the superadditivity of the function f (see Proposition 1.1) the set of all G_b -cells in Δ can be replaced by the set of G_b -cells with labels of the form $v(Y_1, \dots, Y_k)$ and with $\sum |Y_{ij}|$ at most $f(O(n^4))$.

By Lemma 2.3 every G_b -cell Π labeled by a word $v(Y_1, \dots, Y_k)$, can be replaced by a some number N_Π of cells corresponding to relations (2.1) - (2.14). The straightforward computation shows that $N_\Pi = O(|Y_1| + \dots + |Y_k|)^2$. Therefore the set of all cells corresponding to relations (2.1) - (2.14), that replace all G_b -cells of Δ is $O(f(O(n^4))^2)$.

Finally, every disc which is not a hub, can be replaced by a hub and several cells corresponding to relations (2.1) - (2.6). We need at most $O(l^2)$ such cells for every disc of perimeter ℓ which can be easily derived from the proof of Lemma 2.2. Therefore all discs in Δ can be replaced by $O(n^4)$ cells of types (2.1) - (2.5).

Summing all the upper bounds for the numbers of cells of different types, we obtain an isoperimetric function of the group $H(v, m)$ that is equal to $O(f(O(n^4))^2)$. To achieve the better upper bound of $n^2 O(f(n^2)^2)$ claimed in the Theorem, we prove that the perimeter of every G_b -cell in Δ is in fact bounded by $O(n^2)$. The explanation is completely similar to the proof of Lemma 5.15 in [2]. It is based on the fact that the number of the maximal b -bands starting on the same G_b -cell and terminating on a -cells lying in the same a -band, cannot exceed 3 (see Lemma 4.34 in [2]). We leave details to the reader. \square

Proof of Theorem 1. \square The first statement of the theorem follows from Lemmas 2.3, 4.4 and 6.1. To prove that the constructed embedding of G into $H(v, m)$ is undistorted, we need a longer and more complicated argument. The reasoning essentially coincides with that in Sections 10-12 of [12] or in Section 7 in [2]. Therefore we will not repeat it here referring the reader to [12] and [2]. \square

7 The embeddings of the groups $BS_{k,1}$

It is more convenient to change the names of generators of $BS_{k,1}$ to b_1, b_2 . So $BS_{k,1} = \langle b_1, b_2 \mid b_2^{b_1} = b_2^k \rangle$. Obviously the words $W_n = W_n(b_1, b_2) \equiv (b_1^n b_2 b_1^{-n}) b_2 (b_1^n b_2^{-1} b_1^{-n}) b_2^{-1}$ represent the identity in the group $B = BS_{k,1}$, and it is known that their “areas” exponentially increase depending on n if $k \geq 2$. To prove Theorem 2 we are going to embed $BS_{k,1}$ into a finitely presented group $H = H_{k,1}$ such that areas of the words W_n with respect of defining

relations of H have quadratic growth (a relatively easier task), and then we will prove a polynomial isoperimetric inequality for all words vanishing in H (a harder job).

The embedding is similar to the one used for relatively free groups (see the proof of Theorem 1 above). But in order to show flexibility of our approach, we modify the second step of the embedding slightly. The main difference of the construction that we are about to present and the construction presented above is the absence of the letter d . Instead, we have different letters in different sectors of the discs.

An auxiliary group $G = G_{k,1}$ is given by generators

$$a_1, a_2, c, q_1, q_2, q_3, q_4, k_1, \dots, k_N$$

where $N \geq 29$ and by defining relations

$$rq_1r^{-1} = q_1a_1, rq_2r^{-1} = q_2a_1^{-1}, rq_3r^{-1} = q_3a_1, rq_4r^{-1} = q_4a_1^{-1}, rq_5r^{-1} = q_5c; \quad (7.1)$$

$$rk_jr^{-1} = k_j, \quad 1 \leq j \leq N; \quad (7.2)$$

$$ra_1r^{-1} = a_1, ra_2r^{-1} = a_2, rcr^{-1} = c; \quad (7.3)$$

$$q_1a_2q_2a_2q_3a_2^{-1}q_4a_2^{-1}k_1q_5 \dots k_Nq_5 = 1. \quad (7.4)$$

The relation (7.4) is called the *hub* relation.

Note that it is easy to draw the disc corresponding to these relations and write the rules of the corresponding S -machine. It is a good exercise for a reader who wants to learn how to draw van Kampen diagrams and write programs for S -machines.

The words $\Sigma_s = \Sigma(a_1^s, a_2)$ are defined as follows:

$$\Sigma_s \equiv q_1a_1^sa_2q_2a_1^{-s}a_2q_3a_1^sa_2^{-1}q_4a_1^{-s}a_2^{-1}k_1q_5c^s \dots k_Nq_5c^s.$$

An obvious analog of Lemma 2.2 says that $r\Sigma_sr^{-1} = \Sigma_{s+1}$ in view of relations (7.1) – (7.3), and the deduction takes $O(n^2)$ of application of relations (7.1) – (7.3).

The group $H = H_{k,1}$ is defined by adding to the presentation of the group G new generators b_1, b_2, ρ , and by adding new relations

$$\rho a_j \rho^{-1} = a_j b_j, \quad j = 1, 2; \quad (7.5)$$

$$\rho c \rho^{-1} = c; \quad (7.6)$$

$$\rho q_j \rho^{-1} = q_j, \quad j = 1, 2, 3, 4, 5; \quad (7.7)$$

$$\rho k_j \rho^{-1} = k_j, \quad 1 \leq j \leq N; \quad (7.8)$$

$$b_j a_i = a_i b_j, \quad i, j = 1, 2; \quad (7.9)$$

$$b_j q_i = q_i b_j, \quad i = 1, 2, 3, 4, j = 1, 2; \quad (7.10)$$

$$b_1 b_2 b_1^{-1} = b^k. \quad (7.11)$$

By definition, a B -cell in a diagram over H corresponds to any cyclically reduced word $w(b_1, b_2)$ such that $w = 1$ follows from (7.11). The following analog of Lemma 2.3 can be verified immediately (and similar to the proof of Lemma 2.3).

Lemma 7.1 *For any $n \geq 0$ the relation*

$$\Sigma_n \equiv q_1 a_1^n a_2 q_2 a_1^{-n} a_2 q_3 a_1^n a_2^{-1} q_4 a_1^{-n} a_2^{-1} k_1 q_5 c^n \dots k_N q_5 c^n = 1$$

can be obtained by application of $O(n^2)$ relations (7.1) – (7.3) to the hub, and the relation $W_n(b_1, b_2) = 1$ can be deduced from (7.1) – (7.10) in $O(n^2)$ steps.

□

The proof of the next claim is absolutely similar to the proof of Lemma 3.1.

Lemma 7.2 *A minimal hub-free diagram over the presentation of the group $H_{k,1}$ has no ρ -, r -, (r, q) -, q -, (r, k) -, k -, (ρ, k) -, (a, b) -, (ρ, a) -, (ρ, c) -, (r, a) -, (r, c) -, b -, a -, (ρ, q) - or (q, b) -annuli.*

□

The statement of Lemma 4.1 is also true for the group $G = G_{k,1}$, because the boundary label of a spoke has the form r^ℓ , but in the HNN-extension of a free group with the stable letter r defined by relations (7.1)–(7.3), the word r^ℓ can commute with a subword of the left-hand side of (7.4) written between neighbor k -letters only if $\ell = 0$. In view of minimality of Δ , this means that the spokes have length 0, and the two hubs form a mirror pair of cells, that cancel.

Lemma 7.3 *The statement of Lemma 4.2 is true for diagrams over $H = H_{k,1}$. Hence Lemmas 4.3, 4.4 are also valid for H .*

□ As in the proof of Lemma 4.2, we have to consider the boundary label $V = V(\rho^{\pm 1}, r^{\pm 1})$ of the spokes T_1, T_2 that must commute with q_5 in view of relations (7.1)–(7.3), (7.4)–(7.11). This is possible if and only if the exponent sum for r is zero in V , since the group defined by (7.1)–(7.3), (7.4)–(7.11) has the retraction preserving r, q_5 and c and mapping the other generators to 1.

As in Lemma 4.2 we have to prove that V commutes with the cyclic permutation W of the left-hand side of (7.4) beginning with k_j . The word V commutes with any letter

k_j by relations (7.2), (7.8). Therefore it suffices to prove that V commutes with the word $\Lambda_0 \equiv q_1 a_2 q_2 a_2 q_3 a_2^{-1} q_4 a_2^{-1}$.

It is clear that the equality $r\Lambda_n r^{-1} = \Lambda_{n+1}$ follows from (7.1)–(7.3) for $\Lambda_s \equiv q_1 a_1^s a_2 q_2 a_1^{-s} a_2 q_3 a_1^s a_2^{-1} q_4 a_1^{-s} a_2^{-1}$ and any integer n . The equalities $\rho\Lambda_s \rho^{-1} = \Lambda_s$ follows from (7.5), (7.7), (7.9)–(7.10) and Lemma 7.1. Since the exponent sum for the occurrences of r in V is equal to 0, $V\Lambda_0 V^{-1} = V_0$, as desired. \square

Let us mention another similarity of the groups $H(v, m)$ and $H_{k,1}$: the claims of Lemmas 5.1–5.4 are true for $H = H_{k,1}$ as well, and the proofs are quite analogous to those in Section 5.

To complete the proof of Theorem 2 we need

Lemma 7.4 *Let $w = w(b_1^{\pm 1}, b_2^{\pm 1})$ be a word of length n such that $w = 1$ in H . Then this equality can be derived from the trivial one by application of $O(n^4)$ of relations (7.1)–(7.11).*

\square Since $w = 1$ in $BS_{k,1}$, the exponential sum over the occurrences of b_1 in w is equal to 0. Therefore the word w is freely equal to a product $\prod_i v_i$ where $v_i \equiv b_1^{s_i} b_2^{\pm 1} b_1^{-s_i}$, $|s_i| < n/2$ and the number of factor do not exceed n . Passing to a freely conjugate word, we may assume that $0 \leq s_i < n$ for every i .

Let $s = \min s_i$, and assume there are factors $v_i \equiv b_1^s b_2 b_1^{-s}$ and $v_j \equiv b_1^s b_2^{-1} b_1^{-s}$. Then by Lemma 7.1 we can transpose v_i with a neighbor factor $v_{i\pm 1}$ by applying the defining relations at most $O(n^2)$ times, and so after $O(n^3)$ applications of the relations the factors v_i and v_j cancel.

Now assume that all the v_i 's with $s_i = s$ are equal, and ℓ is the number of such v_i 's. Since $b_1^t b_2 b_1^{-t} = b_2^t$ for any $t \geq s$ in G , and the product of the words v_i is equal to 1 in G , the number ℓ is a multiple of k .

Again, by Lemma 7.1 we can collect the $\ell = km$ factors (with minimal s_i 's) at the end of the word w applying the relations at most $\ell O(n^3)$ times. This suffix is (in the free group) the product of m factors $u_i \equiv b_1^s b_2^{\pm k} b_1^{-s}$, and applying relator (7.11) m times we can rewrite it as the product of m factors $b_1^{s+1} b_2^{\pm 1} b_1^{-s-1}$.

Thus, we need $O(n^3)$ relations to reduce the word w to 1, or $\ell O(n^3)$ relations to decrease the number of factors v_i by $\ell(k-1)$. The lemma is proved. \square

Now an isoperimetric inequality for the group $H = H_{k,1}$ can be obtained by almost the same reasoning as in Lemma 6.1 for $H = H(v, m)$ with the only essential difference that the function $f(n^2)^2$ should be replaced by $(n^2)^4$, since by Lemma 7.4 the “area” of a B -cell of perimeter s (in the relators (7.1)–(7.11)) is at most $O(s^4)$. The property that the constructed in the proof of Theorem 2 embedding is undistorted, can be justified in the same manner as for Theorem 1.

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